

Topological Equivalence and Exponential Dichotomy of Linear Impulsive Equations

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The notions of kinematical similarity and topological equivalence for abstract impulsive differential equations are introduced and some of their properties related to the notion of exponential dichotomy are investigated.

1. INTRODUCTION

The work of Mil'man and Myshkis (1960) set the beginning of the qualitative investigation of impulsive equations. Subsequent work devoted to this subject includes Bainov *et al.* (1988*a-d*) and Palmer (1979*a*).

In the present paper the dependence between topological equivalence and exponential dichotomy of impulsive equations is investigated and some results of Palmer (1979*a,b*), Coppel (1978), and Daleckii and Krein (1974) are generalized.

2. STATEMENT OF THE PROBLEM

Let X be an arbitrary Banach space with identity operator I . By $L(X)$ we denote the space of all linear bounded operators acting in X .

Let \mathcal{J} be an arbitrary interval on the real axis $\mathbb{R} = (-\infty, \infty)$.

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Consider the impulsive equation

$$\frac{dx}{dt} = A(t)x \quad (t \neq t_n) \quad (1)$$

$$x(t_n^+) = Q_n x(t_n) \quad (2)$$

where the sequence of points $T = \{t_n\}_{n=-\infty}^{\infty}$ satisfies the condition

$$t_n < t_{n+1} \quad (n = \pm 1, \pm 2, \pm 3, \dots), \quad \lim_{n \rightarrow \pm \infty} t_n = \pm \infty$$

Set $N_{\mathcal{J}} = \{k: t_k \in \mathcal{J}\}$.

We shall say that condition (A) is met if the following conditions hold:

A1. The operator-valued function $A(\cdot): \mathcal{J} \rightarrow X$ is continuously extendable on each interval $[t_n, t_{n+1}]$.

A2. $Q_n \in L(X)$ ($n \in N_{\mathcal{J}}$).

Definition 1. A solution of the impulsive equation (1), (2) we shall call a function $\varphi(t)$ which is continuous for $t \neq t_n$, had discontinuities of the first kind at $t = t_n$, is continuous from the left, for $t \neq t_n$ satisfies equation (1), and at $t = t_n$ meets the condition of a "jump" (2).

Definition 2. An operator $U(t, \tau)$ associating with each element $\xi \in X$ a solution $x(t) = U(t, \tau)\xi$ of equation (1), (2) for which $x(\tau) = \xi$ is said to be a *fundamental operator* of (1), (2). Instead of $U(t, 0)$ we shall write $U(t)$.

The validity of the following formulas is immediately established:

$$U(t, t) = I, \quad U(t, s) = U(t, \tau)U(\tau, s) \quad (3)$$

$$U'_t(t, \tau) = A(t)U(t, \tau) \quad (t \neq t_n), \quad U'_\tau(t, \tau) = U(t, \tau)A(\tau) \quad (\tau \neq t_n) \quad (4)$$

$$U(t_n^+, \tau) = Q_n U(t_n, \tau) \quad (5)$$

where $s \leq \tau \leq t$.

Lemma 1. Let conditions (A) hold.

Then for $\tau \leq t < \infty$ ($t, \tau \in \mathcal{J}$) the following formula is valid:

$$U(t, \tau) = \begin{cases} U_0(t, \tau), & t_n < \tau \leq t \leq t_{n+1} \\ U_0(t, t_n) \left(\prod_{j=n}^{k+1} Q_j U_0(t_j, t_{j-1}) \right) Q_k U_0(t, \tau) & t_{k-1} < \tau \leq t_k < t_n < t \leq t_{n+1} \end{cases}$$

where $U_0(t, \tau)$ is the fundamental operator of the equation

$$\frac{dx}{dt} = A(t)x \quad (6)$$

We shall say that condition (B) is met if the operators Q_n have bounded inverse ones.

Lemma 2. Let conditions (A) and (B) hold. Then for $\tau, t \in \mathcal{I}$ the following formula is valid:

$$U(t, \tau) = \begin{cases} U_0(t, \tau), & t_n < t \leq \tau \leq t_{n+1} \\ U_0(t, t_n) \left(\prod_{j=n}^{k+1} Q_j U_0(t_j, t_{j-1}) \right) Q_k U_0(t_k, \tau) & t_{k-1} < \tau \leq t_k < t_n < t \leq t_{n+1} \\ U_0(t, t_n) \left(\prod_{j=n}^{k-1} Q_j^{-1} U_0(t_j, t_{j+1}) \right) Q_k^{-1} U_0(t_k, \tau) & t_{n-1} < t \leq t_n < t_k < \tau \leq t_{k+1} \end{cases}$$

where $U_0(t, \tau)$ is the fundamental operator of (6).

Lemmas 1 and 2 are proved by a straightforward verification.

Lemma 3. Let the function $\varphi(t)$ be piecewise continuous with points of discontinuity τ_n ($n = 1, 2, 3, \dots$) of the first kind and for $t \geq t_0$ let the following inequality hold:

$$\varphi(t) \leq c + \sum_{t_0 < \tau_i < t} a_i \varphi(\tau_i) + \int_{t_0}^t v(\tau) \varphi(\tau) d\tau$$

where $c \geq 0$ and $a_i \geq 0$ are constants and the function $v(t) > 0$ is locally summable on the half-axis $t_0 \leq t < \infty$.

Then the following estimate is valid:

$$\varphi(t) \leq c \prod_{t_0 < \tau_i < t} (1 + a_i) \exp \left[\int_{t_0}^t v(\tau) d\tau \right]$$

The *proof* of the lemma is carried out by means of the lemma of Gronwall and Bellman.

Lemma 4. Let the following conditions hold:

1. Condition (B) is met.
2. The following inequality is valid:

$$\left\| \prod_{j=n}^k Q_j \right\| \leq N e^{-v(t-s)}$$

$$t_{k-1} < s \leq t_k < t_n < t \leq t_{n+1}, \quad v = \text{const}, \quad N > 0, \text{ const}$$

Then for the fundamental operator $U(t, s)$ of the impulsive equation (1), (2) the following estimates are valid:

$$\begin{aligned} N \exp[-v(t-s)] \exp\left[-N \int_s^t \|A(\tau)\| d\tau\right] \\ \leq \|U^{\pm 1}(t, s)\| \leq N \exp[-v(t-s)] \exp\left[N \int_s^t \|A(\tau)\| d\tau\right] \end{aligned} \quad (7)$$

Proof. The fundamental operator of the impulsive equation

$$\frac{dx}{dt} = 0 \quad (t \neq t_n) \quad (8)$$

$$x(t_n^+) = Q_n x(t_n) \quad (9)$$

has the form

$$U_1(t, s) = \prod_{j=n}^{k+1} Q_j \quad (t_{k-1} < s \leq t_k < t_n < t \leq t_{n+1}) \quad (10)$$

Then for $U_1(t, s)$ the following estimate is valid:

$$\|U_1(t, s)\| \leq N e^{-v(t-s)} \quad (11)$$

i.e., the condition of Lemma 4 (Bainov et al., 1988b) is met. For $U(t, s)$ we obtain the estimate

$$\|U(t, s)\| \leq N \exp[-v(t-s)] \exp\left[N \int_s^t \|A(\tau)\| d\tau\right]$$

The estimates

$$\begin{aligned} \exp\left[-v(t-s) - N \int_s^t \|A(\tau)\| d\tau\right] \\ \leq \|U^{\pm 1}(t, s)\| \leq N \exp[-v(t-s)] \exp\left[N \int_s^t \|A(\tau)\| d\tau\right] \end{aligned}$$

are proved in an analogous way. ■

Consider the initial value problem

$$\frac{dx}{dt} = f(t, x) \quad (t \neq t_n) \quad (12)$$

$$x(t_n^+) = Q_n x(t_n) \quad (13)$$

$$x(t_0) = \xi \quad (14)$$

Lemma 5. Let the following conditions hold:

1. The function $f(t, x): \mathcal{J} \times X \rightarrow X$ is continuous on $(\mathcal{J} \setminus T) \times X$ and $f(t_n^-, x) = f(t_n, x)$ ($x \in X$).

2. $\|f(t, x) - f(t, y)\| \leq L(t)\|x - y\|$ ($t \in \mathcal{J}; x, y \in X$).

Moreover, the function $L(t)$ satisfies the condition

$$\tilde{L} = \sup_{a < b < \infty} \frac{1}{b - a} \int_a^b L(s) ds < \infty$$

3. $f(t, 0) = 0$ ($t \in \mathcal{J}$).

4. $Q_n \in L(X)$, $n \in N_{\mathbb{R}}$, $\mathcal{J} = \mathbb{R}$.

Then there exists a unique solution $x(t) = X(t, t_0, \xi)$ of the initial value problem (12)–(14) for $t \in \mathcal{J}$ for which the following estimate is valid:

$$\|x(t)\| \leq \|\xi\| \prod_{t_n \in (\min(t_0, t), \max(t_0, t))} (1 + \|I_i\| e^{L|t - t_0|})$$

$(I_n = Q_n - I)$.

Proof. From the theorems on the existence and uniqueness of solutions of ordinary differential equations (see, e.g., Daleckii and Krein, 1974) there follows the existence of a unique solution $x(t)$ ($t \in \mathcal{J}$) of the initial value problem (12), (13). For the solution $x(t)$ the following representation is valid:

$$x(t) = \begin{cases} \xi + \int_{t_0}^t f(s, x(s)) ds + \sum_{\tau < t_k < t} I_k(x(t_k)), & t \geq t_0 \\ \xi + \int_{t_0}^t f(s, x(s)) ds - \sum_{t \leq t_k < \tau} I_k(x(t_k)), & t < t_0 \end{cases}$$

Let $t \geq t_0$. Then

$$\begin{aligned} \|x(t)\| &\leq \|\xi\| + \int_{t_0}^t \|f(s, x(s))\| ds + \sum_{t_0 < t_k < t} \|I_k(x(t_k))\| \\ &\leq \|\xi\| + \int_{t_0}^t L(s)\|x(s)\| ds + \sum_{t_0 < t_k < t} \|I_k\| \cdot \|x(t_k)\| \end{aligned}$$

By Lemma 3 we obtain the estimate

$$\|x(t)\| \leq \|\xi\| \prod_{t_0 < t_k < t} (1 + \|I_i\|) e^{L(t - t_0)} \tag{15}$$

where

$$\tilde{L} = \sup_{a < b < \infty} \frac{1}{b - a} \int_a^b L(s) ds \tag{16}$$

For $t < t_0$ we obtain an estimate analogous to estimate (15). Lemma 5 is proved. ■

3. MAIN RESULTS

Let equations (1), (2) and

$$\frac{dx}{dt} = B(t)x \quad (t \neq t_n) \tag{17}$$

$$x(t_n^+) = \tilde{Q}_n x(t_n) \tag{18}$$

belong to class K .

Definition 3. Equations (1), (2) and (17), (18) are said to be *kinematically similar* on \mathcal{J} if there exists an operator-valued function $S(\cdot)$: $\mathcal{J} \rightarrow L(X)$ for which the following conditions are met:

1. $S(t)$ is bounded and continuous for $t \neq t_n$.
2. $S(t)$ has discontinuities of the first kind at $t = t_n$ and it is continuous from the left.
3. $S(t)$ has a bounded inverse one.
4. If $\varphi(t)$ is a solution of (1), (2), then $S(t)\varphi(t)$ is a solution of (17), (18).

Remark 1. A necessary and sufficient condition for a kinematical similarity of (1), (2) and (17), (18) is the existence of an operator-valued function $S(t)$ ($t \in \mathcal{J}$) with the properties 1–3 of Definition 4 for which one of the following conditions is fulfilled:

$$B = SAS^{-1} + S'S^{-1} \quad (t \neq t_n, t \in \mathcal{J}) \tag{19}$$

$$S(t_n^+)Q_n = \tilde{Q}_n S(t_n) \quad (t_n \in N_{\mathcal{J}}) \tag{20}$$

or

$$S = XY^{-1}$$

where X and Y are some evolutionary operators of equations (1), (2) and (17), (18), respectively.

We shall call the function $S(t)$ a *transforming* one.

Lemma 6. Let the space X be real Hilbert and let it split up into a direct sum of subspaces X_1 and X_2 with the corresponding projectors P_1 and P_2 , i.e., $X_i = P_i X$ ($i = 1, 2$) and $X = X_1 \dot{+} X_2$.

Then there exists a splitting $X = X_1 \dot{+} Y_2$ for which the corresponding projectors $\tilde{P}_1: X \rightarrow X_1$ and $\tilde{P}_2: X \rightarrow Y_2$ are Hermitian.

Proof. Let Y_1 be the orthocomplement to X_1 . Then the projector $\tilde{P}_1: X \rightarrow X_1$ with $\ker \tilde{P}_1 = Y_1$ is Hermitian. The projector $\tilde{P}_2 = I - \tilde{P}_1$ is also Hermitian and X splits up into a direct sum of the subspaces X_1 and $\tilde{P}_2 X$. ■

Lemma 7. Let the space X be Hilbert and let conditions (A), (B) hold. Then equation (1), (2) is kinematically similar to the equation

$$\frac{dx}{dt} = B(t)x$$

where the operator B is Hermitian.

Proof. For the operator A the representation $A = A_R + iA_{\mathcal{I}}$ is valid, where $A_R = (1/2)(A + A^*)$, $A_{\mathcal{I}} = (1/2i)(A - A^*)$. As a transforming function we take the function $V^{-1}(t)$ ($t \in \mathcal{I}$), where $V(t)$ is a solution of the impulsive equation

$$\frac{dV}{dt} = iA_{\mathcal{I}} V \tag{21}$$

$$V(t_n^+) = Q_n V(t_n) \tag{22}$$

Since the operator $iA_{\mathcal{I}}$ is skew-Hermitian, then the function $V(t)$ is unitary and by the Banach–Steinhaus theorem it is bounded. By formula (19) for the operator B we obtain the representation

$$B = V^{-1} A_R V$$

and by formula (20) for the impulsive operators \tilde{Q}_n of the new equation we obtain

$$\tilde{Q}_n = V^{-1}(t_n^+) Q_n V(t_n) = V^{-1}(t_n) Q_n^{-1} Q_n V(t_n) = I$$

Lemma 7 is proved. ■

Theorem 1. Let the following conditions hold.

1. The space X is Hilbert with scalar product (\cdot, \cdot) and splits up into a direct sum of the subspaces X_1 and X_2 with projectors P_1 and P_2 , respectively.

2. Conditions (A) and (B) hold.

3. There exists a constant $M > 0$ for which

$$\|U(t)P_i U^{-1}(t)\| \leq M \quad (i = 1, 2; t \in \mathcal{I})$$

Then the impulsive equation (3), (4) is kinematically similar to the impulsive equation with an operator-valued function $B(t)$ ($t \in \mathcal{I}$) and impulse operators \tilde{Q}_n which commute with the projectors P_i ($i = 1, 2$) and,

moreover, the following estimate is valid:

$$\|B(t)\| \leq \|A(t)\| \quad (t \in \mathcal{J}) \tag{23}$$

Proof. By Lemma 6 we shall assume that the projectors P_1 and P_2 are Hermitian. Set

$$R^2 = P_1 U^*(t)U(t)P_1 + P_2 U^*(t)U(t)P_2 \tag{24}$$

It is not hard to check that the operator $R^2(t)$ ($t \in \mathcal{J}$) is Hermitian and uniformly positive. Let $R(t)$ be the positive square root of $R^2(t)$. The operator $R(t)$ is also Hermitian. Since $R^2(t)$ commutes with P_i ($i = 1, 2$), then $R(t)$ also commutes with P_i . Consider the operator-valued function $S(t) = R(t)U^{-1}(t)$ ($t \in \mathcal{J}$), where $U(t) = U(t, 0)$ is the fundamental operator of (1), (2). A straightforward verification yields that the functions $S(t)$ and $S^{-1}(t)$ ($t \in \mathcal{J}$) are continuous and differentiable for $t \neq t_n$, have discontinuities of the first kind at $t = t_n$, and are continuous from the left. From the equalities

$$\begin{aligned} S^*S &= (U^{-1})^*R^*RU^{-1} = (U^{-1})^*R^2U^{-1} \\ &= (U^{-1})^*P_1U^*UP_1U^{-1} + (U^{-1})^*P_2U^*UP_2U^{-1} \\ &= (UP_1U^{-1})^*(UP_1U^{-1}) + (UP_2U^{-1})^*UP_2U^{-1} \end{aligned}$$

there follows the estimate

$$\|S\|^2 \leq \|UP_1U^{-1}\|^2 + \|UP_2U^{-1}\|^2 \leq M^2 + (1 + M)^2 < \infty$$

We shall show that the function $S^{-1}(t)$ is also bounded

$$\begin{aligned} P_1(S^{-1})^*S^{-1}P_1 + P_2(S^{-1})^*S^{-1}P_2 &= P_1R^{-1}U^*UR^{-1}P_1 + P_2R^{-1}U^*UR^{-1}P_2 \\ &= R^{-1}P_1U^*UP_1P_1R^{-1} + R^{-1}P_2U^*UP_2R^{-1} \\ &= R^{-1}(P_1U^*UP_1 + P_2U^*UP_2)R^{-1} = I \end{aligned}$$

Let $z \in X$ be arbitrarily chosen. Then

$$\begin{aligned} \|S^{-1}z\|^2 &= \|S^{-1}P_1z + S^{-1}P_2z\|^2 \\ &\leq (\|S^{-1}P_1z\| + \|S^{-1}P_2z\|)^2 \\ &\leq 2\|S^{-1}P_1z\|^2 + 2\|S^{-1}P_2z\|^2 \\ &= 2(S^{-1}P_1z, S^{-1}P_1z) + 2(S^{-1}P_2z, S^{-1}P_2z) \\ &= 2(P_1(S^{-1})^*S^{-1}P_1z, z) + 2(P_2(S^{-1})^*S^{-1}P_2z, z) \\ &= 2((P_1(S^{-1})^*S^{-1}P_1 + P_2(S^{-1})^*S^{-1}P_2)z, z) \\ &= 2(z, z) = 2\|z\|^2 \end{aligned}$$

Choose the function $S(t)$ ($t \in \mathcal{J}$) as a transforming one. Then by formula (19) for B we obtain

$$B = SAS^{-1} + S'S^{-1}$$

i.e.,

$$\begin{aligned} B &= RU^{-1}AUR^{-1} + (R'U^{-1} + R(U^{-1})')UR' \\ &= RU^{-1}AUR^{-1} + R'R^{-1} - RU^{-1}U'U^{-1}UR^{-1} \\ &= RU^{-1}AUR^{-1} + R'R^{-1} - RU^{-1}AUU^{-1}UR^{-1} \\ &= R'R^{-1} \end{aligned}$$

For the impulsive operators \tilde{Q}_n by (20) we obtain

$$\begin{aligned} \tilde{Q}_n &= S(t_n^+)Q_nS^{-1}(t_n) \\ &= R(t_n^+)U^{-1}(t_n)Q_n^{-1}U(t_n)R^{-1}(t_n) \\ &= R(t_n^+)U^{-1}(t_n)Q_n^{-1}Q_nU(t_n)R^{-1}(t_n) \\ &= R(t_n^+)R^{-1}(t_n) \end{aligned}$$

The fundamental operator of the new equation is $\tilde{U} = SU = R$. It is not hard to check that the operators $B(t)$ and \tilde{Q}_n commute with P_1 , hence with P_2 , too. We shall prove inequality (23). We differentiate equality (24) and obtain

$$RR' + R'R = P_1U^*(A^* + A)UP_1 + P_2U^*(A^* + A)UP_2$$

For an arbitrary $z \in X$ the following equality is valid:

$$((RR' + R'R)z, z) = ((A^* + A)UP_1z, UP_1z) + ((A^* + A)UP_2z, UP_2z) \quad (25)$$

For any $t \in \mathcal{J}$ there exists a minimal constant $\beta(t)$ for which for $u \in X$ the following inequality is valid:

$$([A^*(t) + A(t)]u, u) \leq \beta(t)(u, u)$$

Then from (25) we obtain

$$((RR' + R'R)z, z) \leq \beta\{(UP_1z, UP_1z) + (UP_2z, UP_2z)\} \quad (z \in X)$$

i.e.,

$$((RR' + R'R)z, z) \leq \beta(R^2z, z) \quad (26)$$

Set $V = Rz$. Then from (26) it follows that

$$((RR' + R'R)R^{-1}v, R^{-1}v) \leq \beta(Rv, R^{-1}v)$$

i.e.,

$$((R'R^{-1} + R^{-1}R')v, v) \leq \beta(v, v)$$

hence

$$((R'R^{-1} + R^{-1}R')v, v) \leq \beta \|v\|^2 \quad (27)$$

Hence the operator

$$\frac{1}{2}(R'R^{-1} + R^{-1}R') = \frac{1}{2}(B + B^*) = B_R$$

is bounded and from (27) it follows that $\|B_R\| \leq \|A\|$. Inequality (23) follows from Lemma 7.

Theorem 1 is proved. ■

Definition 4. The impulsive equations (1), (2) and (17), (18) are said to be *topologically equivalent* if there exists a function $h: [0, \infty) \times X \rightarrow X$ with the following properties:

1. $h(t, x) \rightarrow \infty$ as $x \rightarrow \infty$ uniformly in t .
2. The mapping $h_t: X \rightarrow X$ defined by the equality

$$h_t(x) = h(t, x)$$

is a homeomorphism for $t \geq 0$.

3. The mapping $g: [0, \infty) \times X \rightarrow X$ defined by the equality $g(t, x) = (h_t)^{-1}(x)$ enjoys the property 1.

4. If $x(t)$ is a solution of (1), (2), then $h(t, x(t))$ is a solution of (17), (18).

Remark 2. Condition 4 implies the equality

$$h(t, X(t)X^{-1}(s)x) = Y(t)Y^{-1}(s)h(s, x)$$

Remark 3. A straightforward verification shows that the topological equivalence is an equivalence relation in the class of impulsive equations.

Remark 4. If equations (1), (2) and (17), (18) are kinematically similar, then they are topologically equivalent. Indeed, in the definition of topological equivalence it suffices to set $h(t, x) = S(t)x$, where $S(t)$ is the function realizing the kinematical similarity.

Let $Y \subset X$ be an arbitrary subspace with projector P_1 ($Y = P_1X$) and let $P_2 = I - P_1$.

Definition 5. Y is said to *induce an exponential dichotomy* for equation (1), (2) if the following inequalities are valid:

$$\begin{aligned} \|U(t)P_1U^{-1}(\tau)\| &\leq M e^{-\delta(t-\tau)} & (\tau < t < \infty) \\ \|U(t)P_2U^{-1}(\tau)\| &\leq M e^{-\delta(\tau-t)} & (t < \tau < \infty) \end{aligned}$$

where M, δ are positive constants. In this case equation (1), (2) is said to be *exponentially dichotomous* (with projectors P_1 and P_2).

Let Y be an arbitrary Banach space. By $B(Y)$ we shall denote the set of functions $f(\cdot): \mathcal{J} \rightarrow Y$ which are continuous for $t \neq t_n$, have points of discontinuity of the first kind at $t = t_n$, and are continuous from the left, and which are integrally bounded, i.e.,

$$\sup_{t \in \mathcal{J}} \int_t^{t+1} \|f(s)\|_Y ds < \infty$$

The set $B(Y)$ is a Banach space with norm

$$\|f\|_{B(Y)} = \sup_{t \in \mathcal{J}} \int_t^{t+1} \|f(s)\|_Y ds$$

By $M(Y)$ we shall denote the set of sequences $R = \{R_n\}$ whose elements lie in Y and which satisfy the following condition

$$\sup_n \|R_n\| < \infty \tag{28}$$

The set $M(Y)$ is a Banach space with norm

$$\|R\|_{M(Y)} = \sup_n \|R_n\| \tag{29}$$

The space

$$\begin{aligned} \tilde{B} &= B(L(X)) \times M(L(X)) \\ &= \{(A, Q) : A \in B(L(X)), Q \in M(L(X))\} \end{aligned}$$

is Banach with norm

$$\|(A, Q)\|_{\tilde{B}} = \max\{\|A\|_{B(L(X))}, \|Q\|_{M(L(X))}\}$$

Definition 6. The impulsive equation (1), (2) is said to *belong* to \tilde{B} , which is written as $(A, Q) \in \tilde{B}$, if $A \in B(L(X))$ and $Q \in M(L(X))$. If the impulsive equations (A, Q) and (B, \tilde{Q}) are topologically equivalent, we write $(A, Q) \cong (B, \tilde{Q})$.

Definition 7. Equation $(A, Q) \in \tilde{B}$ is said to be *structurally stable* if there exists $\rho > 0$ such that if $\|A - B, Q - \tilde{Q}\|_{\tilde{B}} \leq \rho$, then the impulsive equations (A, Q) and (B, \tilde{Q}) are topologically equivalent.

Lemma 8. The set of structurally stable equations forms an open nonempty subset of the Banach space \tilde{B} . If, moreover, $(A, Q) \in \tilde{B}$ is structurally stable and (A, Q) is kinematically similar to $(B, \tilde{Q}) \in \tilde{B}$, then the equation (B, \tilde{Q}) is also structurally stable.

Proof. Let the equation $(A, Q) \in \tilde{B}$ be structurally stable. Then there exists a number $\rho > 0$ for which the inequality $\|(A, Q) - (B, \tilde{Q})\|_{\tilde{B}} < \rho$ implies $(A, Q) \cong (B, \tilde{Q})$. The set

$$U(B, \tilde{Q}) = \{(C, \tilde{R}) \in \tilde{B}: \|(C, \tilde{R}) - (B, \tilde{Q})\|_{\tilde{B}} < \rho - \|(B, \tilde{Q}) - (A, Q)\|_{\tilde{B}}\}$$

is a neighborhood of (B, \tilde{Q}) in \tilde{B} . Then

$$\|(C, \tilde{R}) - (A, Q)\|_{\tilde{B}} \leq \|(C, \tilde{R}) - (B, \tilde{Q})\|_{\tilde{B}} + \|(B, \tilde{Q}) - (A, Q)\|_{\tilde{B}} < \rho$$

i.e., $(C, \tilde{R}) \cong (A, Q)$. Since $(B, \tilde{Q}) \cong (A, Q)$, then $(B, \tilde{Q}) \cong (C, \tilde{R})$. Hence the equation (B, \tilde{Q}) is structurally stable.

Let the function $S(t)$ realize the kinematical similarity between (A, Q) and (B, \tilde{Q}) . Since the equation (A, Q) is structurally stable, then there exists a number $\delta > 0$ for which $\|(A, Q) - (F, Z)\|_{\tilde{B}} < \delta$ implies $(A, Q) \cong (F, Z)$. The set

$$U(B, \tilde{Q}) = \left\{ (C, L) \in \tilde{B}: \|(C, L) - (B, \tilde{Q})\|_{\tilde{B}} < \frac{\delta}{\sup_{t \in \mathcal{J}} \|S(t)\| \sup_{t \in \mathcal{J}} \|S^{-1}(t)\|} \right\}$$

is a neighborhood of (B, \tilde{Q}) in the space \tilde{B} .

Consider the operator-valued function $D(t) = S^{-1}(t)C(t)S(t) - S^{-1}(t)S'(t)$ and let us estimate the expression $\|A - D\|_{B(L(X))}$:

$$\begin{aligned} \|A - D\|_{B(L(X))} &= \|S^{-1}CS - S^{-1}S' - (S^{-1}BS - S^{-1}S')\|_{B(L(X))} \\ &= \|S^{-1}CS - S^{-1}BS\|_{B(L(X))} \\ &\leq \|S^{-1}\|_{B(L(X))} \|CB\|_{B(L(X))} \|S\|_{B(L(X))} < \delta \end{aligned}$$

The equation (C, L) is kinematically similar to the equation (D, R) , where $R\{R_n\}$, $R_n = S^{-1}(t_n^+)L_nS(t_n)$, hence $(C, L) \cong (D, R)$. Let us estimate $\|R - Q\|_{M(L(X))}$:

$$\begin{aligned} \|R - Q\|_{M(L(X))} &= \sup_n \|R_n - Q_n\|_{L(X)} \\ &= \sup_n \|S^{-1}(t_n^+)L_nS(t_n) - S^{-1}(t_n^+)\tilde{Q}_nS(t_n)\|_{L(X)} \\ &\leq \sup_{t \in \mathcal{J}} \|S^{-1}(t)\| \sup_{t \in \mathcal{J}} \|S(t)\| \sup_n \|L_n - \tilde{Q}_n\|_{L(X)} \\ &= \sup_{t \in \mathcal{J}} \|S^{-1}(t)\| \sup_{t \in \mathcal{J}} \|S(t)\| \cdot \|L - \tilde{Q}\|_{M(L(X))} < \delta \end{aligned}$$

From the estimate obtained it follows that $\|(D, R) - (A, Q)\|_{\tilde{B}} < \delta$ and $(D, R) \cong (A, Q)$. From the kinematical similarity of (A, Q) and (B, \tilde{Q}) it follows that $(A, Q) \cong (B, \tilde{Q})$.

Lemma 8 is proved. ■

Lemma 9. Consider the equation

$$\frac{dx}{dt} = A(t)x, \quad \text{if } t \neq t_n \tag{30}$$

$$x(t_n^+) = Q_n x(t_n) \tag{31}$$

Assume that (A, Q) satisfies the following conditions:

1. Equation (A, Q) satisfies the conditions (A), (B) and has an exponential dichotomy with projector P_1 ; furthermore, $\|A(t)\|$ and $\|Q_n\|$ are bounded.

2. X is a Hilbert space.

3. $\lim_{t \rightarrow \infty} [i(t, t + T)/T] = p < \infty$ uniformly in $t \in \mathbb{R}$.

Then equation (30), (31) is topologically equivalent to the standard equation $dx_1/dt = -x_1, dx_2/dt = x_2, x_1 \in X_1, x_2 \in X_2$.

Proof. First we assume $P_1 = 0$.

The proof of this lemma is suggested by Palmer (1979a).

Consider the function

$$V(t, x) = \int_{-\infty}^t \|U(\tau, t)x\|^2 d\tau$$

It is easily shown that $V(t, x)$ enjoys the following properties:

(i) There exist positive constants a, b such that

$$a \|x\|^2 \leq V(t, x) \leq b \|x\|^2$$

(ii) If $x(t)$ is any solution of (30), (31), then $V(t, x(t))$ is continuous with respect to $t \in \mathbb{R}$.

(iii) V is differentiable with respect to x for every fixed t and differentiable with respect to t for $t \neq t_n$; furthermore, $dV(t, x(t))/dt = \|x(t)\|^2$, where $x(t)$ is any solution of (30), (31).

Consider the function

$$f(t, s, x) = V(t, U(t, s)x)$$

It is seen that f is continuous with respect to t , differentiable with respect to t for $t \neq t_n$; furthermore,

$$\frac{\partial f}{\partial t}(t, s, x) = \|U(t, s)x\|^2$$

Under the assumptions of the lemma we can show without difficulty the existence of positive constants c, d such that

$$c \|x\|^2 \leq \frac{\partial f}{\partial t}(t, s, x) \leq d \|x\|^2 \quad \text{for all } t, s, x$$

Thus the equation

$$1 = f(t, s, x)$$

has a unique solution $t = t(s, x)$ for all $s \in \mathbb{R}, x \in X$; furthermore, t depends on (s, x) continuously for $x \neq 0$.

Now we are in a position to establish the homeomorphism which transforms (30), (31) into the standard equation. In fact, we defined $h: \mathbb{R} \times X \rightarrow \mathbb{R} \times X$ as follows:

$$h(x, x) = h_s(x)$$

where

$$h_s(x) = \begin{cases} e^{s-t(s,x)} \frac{U(t(s,x), s)x}{\|U(t(s,x), s)x\|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It may be noted that h satisfies the conditions 2 and 4 in Definition 5. It is checked that

$$h_s^{-1}(x) = \begin{cases} U(s, u)\{x/[V(u, x)]^{1/2}\} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where

$$u = s - \ln\|x\|$$

Finally, we have to show that

$$\lim_{\|x\| \rightarrow \infty} \|h_t(x)\| = \lim_{\|x\| \rightarrow \infty} \|h_t^{-1}(x)\| = \infty$$

By definition we have

$$\begin{aligned} V(s, x) - 1 &= V(s, x) - V(t(s, x), X(t(s, x), s, x)) \\ &= \int_{t(s, x)}^s D_- V(u, X(u, s, x)) du \end{aligned}$$

where

$$\begin{aligned} D_- V(t, X(t, s, x)) &= \lim_{k \rightarrow +0} \frac{1}{k} [V(t_n, x(t_n)) - V(t_n - k, x(t_n - k))] \\ x(t) &= X(t, s, x) \end{aligned}$$

Hence

$$|s - t(s, x)| \leq |V(s, x) - 1| \left/ \inf_{\substack{u \in [t(s, x), s] \\ u \neq t_n}} \frac{\partial f}{\partial u}(u, s, x) \right.$$

If $\|x\| > 1/\sqrt{a}$, we have $V(s, x) \geq 1$. Thus $s \geq t(s, x)$ and we obtain

$$1 \leq f(u, s, x) \leq b \|U(u, s)x\|^2$$

Thus

$$\frac{\partial f}{\partial u}(u, s, x) = \|U(u, s)x\|^2 \geq \frac{1}{b}$$

for $u \in [t(s, x), s]$, $u \neq t_n$. So we get

$$0 \leq s - t(s, x) \leq (V(s, x) - 1)b \leq (a\|x\|^2 - 1)b$$

If $\|x\| \leq 1/\sqrt{b}$, we have $V(s, x) \leq 1$. Then $s \leq t(s, x)$ and we get

$$0 \leq 1 - V(s, x) \leq \bar{c} \|x\|^2 \int_s^{t(s, x)} \exp(K(u - s)) du$$

where \bar{c}, K are defined as positive constants satisfying

$$\|U(t, s)\| \leq \bar{c} \exp(K(t - s)) \quad \text{for all } t \geq s$$

The existence of \bar{c}, K is easily shown by the assumptions of the lemma.

We get

$$0 \leq 1 - b\|x\|^2 \leq \frac{\bar{c}}{K} \|x\|^2 (\exp(K[t(s, x) - s]) - 1)$$

$$\frac{1}{K} \ln \left[\frac{K}{\bar{c} \|x\|^2} \left(1 - b\|x\|^2 + \frac{\bar{c}}{K} \|x\|^2 \right) \right] \leq t(s, x) - s$$

From this and the above arguments it follows that there exists an increasing function $L_1: [0, \infty) \rightarrow [0, \infty)$, with $L_1(0) = 0$, continuous at 0 such that

$$\|h_s(x)\| \leq L_1(\|x\|) \quad \text{for all } s, x$$

Similarly, it is shown that there exists $L_2: [0, \infty) \rightarrow [0, \infty)$ with the same properties as L_1 such that

$$\|h_s^{-1}(x)\| \leq L_2(\|x\|) \quad \text{for all } s, x$$

Finally, from this we deduce without difficulty that

$$\lim_{\|x\| \rightarrow \infty} \|h_s(x)\| = \lim_{\|x\| \rightarrow \infty} \|h_s^{-1}(x)\| = \infty$$

Now we are in a position to prove Lemma 9 without the assumption $P_1 = 0$.

From Theorem 1 it follows that (30), (31) is kinematically similar to a reducible equation. Applying repeatedly the above result we obtain the proof of Lemma 9. ■

Theorem 2. Let the conditions of Lemma 9 hold. Then equation (A, Q) is structurally stable.

Proof. We complete the definition of the function $A(t)$ ($t \in \mathcal{J}$) on \mathbb{R} in such a way that the new function $\tilde{A}(t)$ is bounded and the equation

$$\frac{dx}{dt} = \tilde{A}(t)x \quad (t \neq t_n) \quad (32)$$

$$x(t_n^+) = Q_n x(t_n) \quad (t_n \in \mathcal{J}) \quad (33)$$

is exponentially dichotomous on $\mathbb{R} \setminus \mathcal{J}$ with projector P_2 for which $\ker P_2 + \text{Im } P_1 = X$. Then (Coppel, 1978) equation (32), (33) will be exponentially dichotomous on \mathbb{R} with projector P , $\ker P = \ker P_2$, $\text{Im } P = \text{Im } P_1$. By Lemma 9, equation (32), (33) is topologically equivalent on \mathbb{R} to the system

$$\frac{dx_1}{dt} = -x_1 \quad (x_1 \in X_1) \quad (34)$$

$$\frac{dx_2}{dt} = x_2 \quad (x_2 \in X_2) \quad (35)$$

If we consider the restrictions of all functions of the system (34), (35) on \mathcal{J} , we obtain that $(A, Q) \cong (34), (35)$.

By Theorem 3 (Bainov et al., 1989) there exists a number $\delta > 0$ such that for $\|(B, R) - (A, Q)\|_B < \delta$, equation (B, R) is exponentially dichotomous with subspaces Y_1 and Y_2 . From Lemma 9 it follows that (B, R) is topologically equivalent to the system

$$\frac{dx_1}{dt} = -x_1 \quad (x_1 \in Y_1) \quad (36)$$

$$\frac{dx_2}{dt} = x_2 \quad (x_2 \in Y_2) \quad (37)$$

Moreover, (34), (35) \cong (36), (37); therefore $(B, R) \cong (34), (35)$, i.e.,

$$(A, Q) \cong (B, R)$$

Theorem 2 is proved. ■

Remark 5. Following the ideas of the exposition in Palmer (1979b) and Lemmas 4 and 5, it is not hard to check that for $\dim X < \infty$ if equation (A, Q) is structurally stable, then it is exponentially dichotomous, too.

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